Generalization of symmetry

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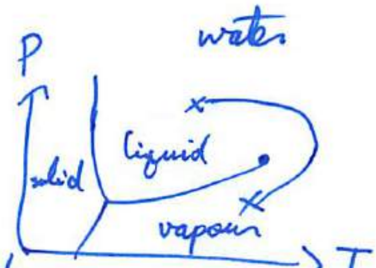
1. Motivation
2. Reminder of ordinary symmetry
3. Generalization of symmetry : 1-form symmetry.

(Ref. Generalized global symmetries, Gaiotto, Kapustin, Seiberg, Willet)

1. Motivati

"Phases of matters".



- Two thermal states are in the same phase

$\Longleftrightarrow$ One can connect those states continuously without phase transitions.

ex) Liquid and vapour of water are in the same phase. (van der Waals)

I would say this is "rigorous" but not necessarily "practical".

- To judge two states are different as phases, you must check ALL possible continuous paths connecting them.

It sounds almost hopeless to judge two states are different.

Symmetry

Landau criterion:

If SSB pattern $G \to H$ are different, the two states are different.

Q.) Can the converse of Landau criterion be also true?

A.) (Mostlikely) yes, in classical statistical physics.

But, in quantum many-body systems, No!

counter example) · Confinement / Higgs phase transitions in gauge theories
· Fractional QHE. ...

(Wilson)

$SU(N)$ YM + adj. matters.

Assume mass gap $\Delta > 0$.

Confinement / Deconfinement is distinguished by the behavior of the Wilson loop

$$W(C) = \frac{1}{N} \mathrm{tr}\left(\mathcal{P} \exp i \oint_C a\right).$$

For large loops C,

$$W(C) \sim \begin{cases} \exp(-\sigma \cdot \mathrm{Area}(C)) & \text{(Area law)} \\ 1 & \text{(perimeter law)} \end{cases}$$

Wilson's proposal :

$$\left[\begin{array}{l} \text{Confinement} \Longleftrightarrow \text{Area law.} \\ \text{Deconfinement} \Longleftrightarrow \text{Perimeter law.} \end{array}\right.$$

Distinction of phases not by $\langle \mathcal{O}(x) \rangle \underset{=0}{\neq 0}$.

In condensed-matter terminology, this means.

$$\begin{cases} \text{Confinement} \Leftrightarrow \text{Trivial topological order (i.e. No top. order)} \\ \text{Deconfinement} \Leftrightarrow \mathbb{Z}_N \text{ topological order.} \end{cases}$$

Top. order : # of G.S. depends on the top. of spatial manifolds.

$$M_4 = \underset{\text{space}}{T^3} \times \underset{\text{time}}{\mathbb{R}} \ , \ \underset{\text{space}}{S^3} \times \underset{\text{time}}{\mathbb{R}}$$

On T^3, we can consider Polyakov loop (i.e. Wilson loop for noncontractable loop)

$$P_i = \frac{1}{N} \mathrm{tr} \left(\mathcal{P} \exp \left(i \int_0^{L_i} a_i \, dx_i \right) \right) \qquad (i=1,2,3).$$

"Center sym." Under "aperiodic" gauge trans. $P_i \to e^{\frac{2\pi i}{N}} P_i$.

In confined phase $\langle P_i \rangle = 0$.

$$\left(\begin{array}{l} \Rightarrow \# (\text{G.S. on } T^3) = 1 \\ \text{Also for } S^3 \text{, no contractable loop exists, so} \\ \qquad \# (\text{G.S. on } S^3) = 1 \end{array} \right).$$

In deconfined phase, $\langle P_i \rangle = \# e^{\frac{2\pi i}{N} k_i}$ $(k_i = 1, \cdots, N)$.

$$\Rightarrow \# \text{ G.S. on } T^3 = N^3$$

On the other hand, $\Big\}$ different.

$$\# \text{ G.S. on } S^3 = 1.$$

Q.) Is top. order an order for some sym.?

What do we really mean by center sym.?

2. Reminder of ordinary symmetry

We here present the definition of ordinary symmetry, (which we'll later call 0-form sym.). We'll generalize it to p-form symmetry later.

Assume we have an action $S[\phi]$ (although this is not necessary)

We have symmetry G, if

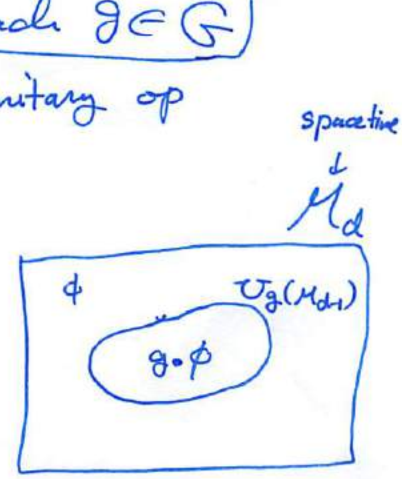
$$\left(\begin{array}{l} \phi \longrightarrow g\cdot\phi \quad \text{for } g\in G, \\ g\cdot\phi \neq \phi \quad \text{if } g\neq 1\in G, \\ S[g\cdot\phi] = S[\phi] \end{array} \right.$$

Let's translate these into more abstract language, which turns out to be useful ~~for generalizations~~.

$\boxed{\text{Symmetry} \iff \text{Top. defect. on co-dim 1 surface for each } g\in G}$

$\phi \to g\cdot\phi \iff$ We have some unitary op

$$U_g(M_{d-1})$$



s.t.

$$U_g(S^{d-1}_x)\,\phi(x) = g\cdot\phi(x)$$

$S[g\cdot\phi] = S[\phi] \iff$ We can deform M_{d-1} continuously wo. changing expectation values

$$\langle U_g(M_{d-1})\,\phi\cdots\phi\rangle = \langle U_g(M'_{d-1})\,\phi\cdots\phi\rangle$$

Def (Symmetry)

d-dim. QFT has sym. G.

$\overset{\text{def}}{\Longleftrightarrow}$ X : d-dim. spacetime (Riem.)

$U_g(M_{d-1})$: co-dim 1 defect on $M_{d-1} \subset X$ for $g \in G$.

(Group law)

- $U_{g_1}(M_{d-1})\, U_{g_2}(M_{d-1}) = U_{g_1 g_2}(M_{d-1})$

(Conservation law)

- $U_g(M_{d-1})$ is topological, i.e.

$$\langle U_g(M_{d-1} + \Delta M_{d-1})\ \mathcal{O}(x_1)\cdots\rangle = \langle U_g(M_{d-1})\ \mathcal{O}(x_1)\cdots\rangle.$$

- For local op. $\mathcal{O}_i(0)$

$$U_g(S_0^{d-1})\ \mathcal{O}_i(0) = \underbrace{R(g)_i{}^j}_{\text{rep. of } G}\ \mathcal{O}_j(0).$$

- For some $\mathcal{O}_i \neq 0$, R is faithful rep., i.e. $R \neq 1$ if $g \neq 1$.

//

SSB

We can define SSB of sym. G as follows:

for some $\mathcal{O}_i(x)$ with nontrivial G-rep. R,

$$\langle \mathcal{O}_i(x)\ \mathcal{O}_j^*(0)\rangle \xrightarrow[\substack{|x| \to \infty \\ vol(X) \to \infty}]{} \text{nonzero}.$$

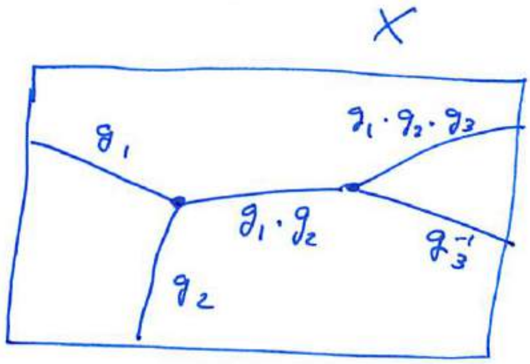
(On compact spacetime, 1-point func. $\langle \mathcal{O}_i \rangle = 0$.
So, we define SSB as off-diagonal long-range order.)

Gauging G

Here, we assume G is a discrete group like $\mathbb{Z}_N$, S_N, etc.

How do we gauge G?

We consider a network of top. defects $U_{g_i}(M_{d-1})$.



$\langle O_1 \cdots O_n \rangle_{\text{gauged}}$ $\quad \hookrightarrow e^{i S_{\text{top.}}(\text{network})}$

$$:= \sum_{\text{network}} e^{i S_{\text{top.}}(\text{network})} \langle \underbrace{\prod_i U_{g_i}} \; O_1 \cdots O_n \rangle$$

insertion of top. defects, corresponding to the network.

This indeed gives the projection to G-singlet states! (Elitzur)

O_i : some non-trivial rep. of G $*1$

$$\langle O_i(0) \cdots \rangle_{\text{gauged}} = \langle \overbrace{U_g(S^{d-1}_0)\, O_i(0)}^{R_g \cdot O_i(0)} \cdots \rangle_{\text{gauged}}$$

$$= R_g \cdot \langle O_i(0) \cdots \rangle_{\text{gauged}}.$$

$$\Rightarrow \langle O_i(0) \cdots \rangle_{\text{gauged}} = 0.$$

//

3. Generalized symmetry

We now consider generalizati. of symmetry.

In our def., sym. is generated by

$U_g(M_{d-1})$: topological codim-$\underline{\underline{1}}$ op.

for $g \in \underset{\text{group}}{\underline{\underline{G}}}$.

Does it have to be $\underline{\underline{1}}$?

Does U have to be associated to group?

I don't talk about it (Ref. Bhardwaj, Tachikawa 2017)

Higher-form symmetry

Def (p-form sym)

d-dim QFT has p-form sym. G

$\overset{\text{def}}{\Longleftrightarrow}$ X: d-dim. spacetime

$U_g(M_{d-p-1})$: codim-$(p+1)$ defect

- $U_{g_1}(M_{d-p-1})\, U_{g_2}(M_{d-p-1}) = U_{g_1 g_2}(M_{d-p-1})$.
- $U_g(M_{d-p-1})$ is topological.
- $\mathcal{O}(C^{(p)})$: extended objects defined on p-dim. closed mfd $C^{(p)} \subset X$.

$$U_g(S^{d-p-1})\, V(C^{(p)}) = R(g) \cdot V(C^{(p)})$$

- For some V, R is faithful. //

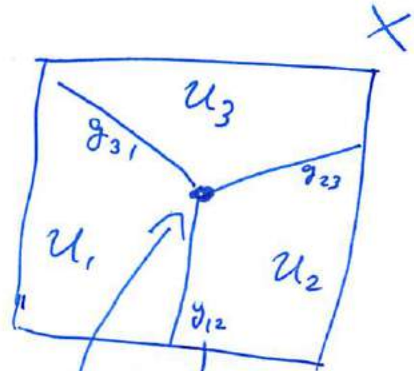
(Nontrivial mixture of different p-form sym. (such as mixture of 0-form $\times$ 1-form) is possible $\Rightarrow$ n-group sym. (Kapustin, Thorngren 2013, Cordova, Dumitrescu, Intriligator 2018))

We can play with SSB, gauging of p-form sym.!!

Especially, "center sym" = $\mathbb{Z}_N$ 1-form symmetry.
(cf. Any p-form sym ($p \geq 1$) is Abelian, thus $G = U(1)^r \times \mathbb{Z}_{n_1} \times \cdots$).

$$\begin{cases} \text{Area law} = \text{Unbroken } \mathbb{Z}_N \text{ 1-form} \\ \text{Perimeter law} = \underbrace{\text{SSB of } \mathbb{Z}_N \text{ 1-form}} \end{cases}$$
$\hookrightarrow \mathbb{Z}_N$ TQFT.

Let's check explicitly that $SU(N)$ pure YM has $\mathbb{Z}_N$ 1-form sym. : $W(C) \longmapsto e^{\frac{2\pi i}{N}} W(C)$.



SU(N) gauge field a

= Collection of

1-form Lie(SU(N))-valued fields

a_i on $\mathcal{U}_i$

with connection formula

$$a_j = g_{ij}^{-1} a_i g_{ij} + g_{ij}^{-1} d g_{ij} \quad \text{—— } (*)$$

transition function. $\in SU(N)$

On $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$,

g_{ij}'s must satisfy

$$g_{ij} g_{jk} g_{ki} = 1 \quad \text{—— } (**)$$

We can have co-dim 2 defect $U_{e^{\frac{2\pi i}{N}}}(\underbrace{\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k}_{M_{d-2}})$,

s.t. we instead require

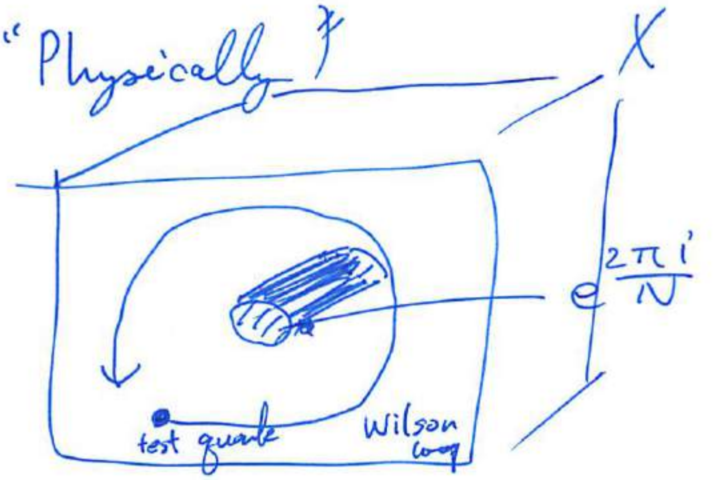
$$g_{ij} g_{jk} g_{ki} = e^{\frac{2\pi i}{N}} \quad \leftarrow \text{'t Hooft magnetic flux.}$$

For other $\mathcal{U}_\ell \cap \mathcal{U}_m \cap \mathcal{U}_n$, we require $(**)$.

This does not change $(*)$ at all, so the Boltzmann weight $e^{-S_{YM}[a]}$ is unchanged.

$\rightarrow U_{e^{\frac{2\pi i}{N}}}(M_{d-2})$ is topological.

"Physically"



By this insertion of Aharonov-~~Bohm~~ (AB) phase, Wilson loop detects $e^{\frac{2\pi i}{N}}$, while local operators don't.

$$\Rightarrow \langle W(C)\, U_{e^{\frac{2\pi i}{N}}}(M_{d-2}) \rangle$$
$$= e^{\frac{2\pi i}{N}} \langle W(C) \rangle .$$

<u>Note</u> With fundamental matters ψ, the connection formula $\psi_j = g_{ij}^{-1} \psi_i$ is affected by $U_{e^{\frac{2\pi i}{N}}}(M_{d-2})$.

$\rightarrow$ No $\mathbb{Z}_N$ 1-form sym. with fund. matters. //

Consistent with Fradkin-Shenker complementarity bet. confinement/Higgs phases.

Anomaly with 2-form gauge fields

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Review of 1st lecture

We give an abstract def. of sym: It roughly says

conservati law.

(0-form) Symmetry = Insertion of co-dim.-1 topological defects

$U_g(M_{d-1})$ w./ $\begin{cases} \cdot\, U_{g_1} U_{g_2} = U_{g_1 g_2} \\ \cdot\, U_g(S^{d-1}_0)\, V(0) = R(g)\cdot V(0) \end{cases}$

· for some V, $R \neq 1$ if $g \neq 1$.

⇒ We give a generalized sym (p-form sym) as follows:

p-form sym. = Insertion of co-dim. (p+1) topological defects

$U_g(M_{d-p-1})$

w./ $\begin{cases} \cdot\, U_{g_1} U_{g_2} = U_{g_1 g_2} \\ \cdot\, p\text{-dim. objects } V(C^{(p)}) \text{ transforms as } U_g(M_{d-p-1})\, V(C^{(p)}) = R(g)^{\mathrm{Lk}(M_{d-p-1}, C^{(p)})} \cdot V(C) \end{cases}$

. . .

For 4d SU(N) gauge theory + Adj matters.

fund Wilson loop

$$U_{e^{\frac{2\pi i}{N}}}(M_{d-p-1})\, W(C) = e^{\frac{2\pi i}{N} \mathrm{Lk}(M_{d-p-1}, C)}\, W(C).$$

We'll discuss the new anomaly thanks to this generalization of symmetries.

(cf. Kapustin, Thorngren; Wang, Wen; Gaiotto, Kapustin, Komargodski, Seiberg; Tanizaki, Kikuchi; - - -)

Anomaly, Anomaly matching

Assume we are interested in d-dim. QFT with sym. G.

't Hooft anomaly

(Please don't be confused with Adler-Bell-Jackiw anomaly)

G has an 't Hooft anomaly

$\overset{\text{def}}{\Longleftrightarrow}$ · $Z_{M_d}[A]$: Partition func. on d-dim. mfd M_d with the background G-gauge field A.

Let us consider the G-gauge trans.

$$A \to A + d\lambda \, (+ \underset{O(A\lambda, \lambda^2, \cdots)}{\cdots})$$

then

$$Z_{M_d}[A + d\lambda] = Z_{M_d}[A] \, e^{i \mathcal{A}[A, \lambda]}$$

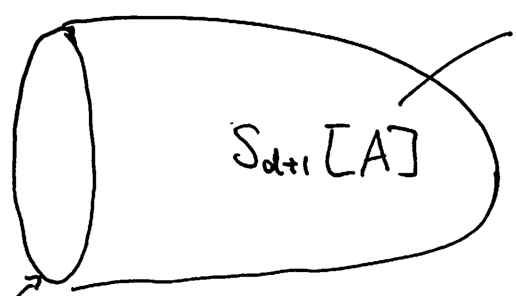
$\mathcal{A}$ is an anomaly, IF we cannot eliminate it by adding d-dim. local counter terms. //

't Hooft anomaly matching

Anomaly $\mathcal{A}$ does not change under the RG flow.

Callan-Harvey mechanism

(d+1)-dim. topological action of A only.



$$\delta_\lambda S_{d+1}[A] = \mathcal{A}[A, \lambda].$$

d-dim QFT

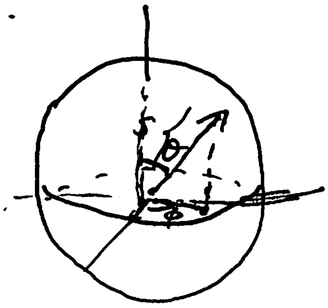
$$\Rightarrow \underbrace{Z_d[A] \, e^{-i S_{d+1}[A]}}$$ is gauge inv.

$\downarrow$ RG flow

$$Z_{d, \text{EFT}}^{\text{IR}}[A] \, e^{-i \underbrace{S_{d+1}[A]}_{\text{top. action cannot be changed}}}$$ is gauge inv. //

"Simplest" example of anomaly, anomaly matching

QM of single spin S w/ $H = J\hat{S}_z^2$.



Lagrangian

$$S_E = i\int \overbrace{S(1-\cos\theta)\,d\phi}^{\text{WZ term}}$$

$$-\left(\int \dot{\vec{n}}^2 + JS^2 \underbrace{n_z^2}_{\cos^2\theta}\right)$$

(WZ term is important : We know $\dim\mathcal{H} = 2S+1$. $\quad S_z = -S, -S+1, \cdots, S$

$p_\phi = S(1-\cos\theta)$. Since $\phi\sim\phi+2\pi$, $p_\phi \overset{n}{=} \in \mathbb{Z}$. $\Updownarrow$

$0 \le 1-\cos\theta = \frac{n}{S} \le 2 \rightsquigarrow n = 0, 1, \cdots, 2S$.)

Spin rotational symmetry $SO(3)$ is explicitly broken down to

$$\underbrace{SO(2)}_{\phi\to\phi+\alpha} \rtimes \underbrace{\mathbb{Z}_2}_{\begin{cases}\phi\to-\phi\\ \theta\to\pi-\theta\end{cases}}$$

This symmetry has 't Hooft anomaly for half-integer spins $S = \frac{1}{2}, \frac{3}{2}, \cdots$.

A: $U(1)$ gauge field

$$S_E = i\underbrace{\int S(1-\cos\theta)(d\phi+A)}_{\downarrow\ \mathbb{Z}_2 \text{ trans}} + \left(|d\phi+A|^2 + \cdots\right)$$

$$i\int S(1+\cos\theta)\{-(d\phi+A)\}$$

$$\Delta S_E = 2iS\int(d\phi+A). \qquad e^{\Delta S_E} = e^{i\int(2S)A}.$$

This looks to be an 't Hooft anomaly of $SO(2) \rtimes \mathbb{Z}_2$:

$$Z[A] \xrightarrow{\mathbb{Z}_2} Z[A]\, e^{i\int (2S) A}.$$

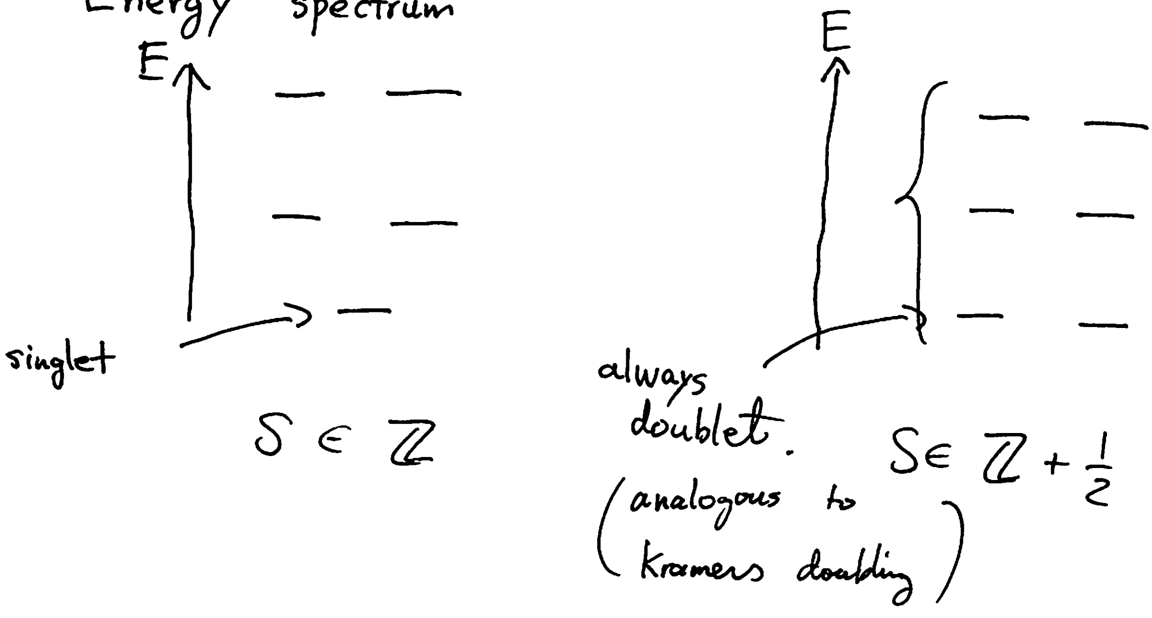
Possible local counter term is $e^{ik\int A}$ (k : integers).

$$Z[A]\, e^{ik\int A} \longmapsto \left(Z[A]\, e^{i\int (2S) A}\right) e^{-i\int k A}$$

$$= \left(Z[A]\, e^{ik\int A}\right) \cdot e^{i\int (2S-2k) A}.$$

- $S = 1, 2, \cdots$: Taking $k = S$, the phase is eliminated.
  $\Rightarrow$ No 't Hooft anomaly
- $S = \frac{1}{2}, \frac{3}{2}, \cdots$: Any $k \in \mathbb{Z}$ cannot eliminate the phase
  $\Rightarrow$ $SO(2) \rtimes \mathbb{Z}_2$ 't Hooft anomaly.

Energy spectrum



Note : This anomaly persists even if we break

$$SO(2) \rtimes \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$$

by adding $\cos(2n\phi)$ as perturbations.
$\sim (\hat{S}_x^2)^n$

2d Maxwell with θ-angle ("Next simplest" example)

We consider

$$S_E = -\frac{1}{2g^2}\int |da|^2 + i\frac{\theta}{2\pi}\int da .$$

$\left((\mathbb{Z}_2)_C \text{ at } \theta = 0, \pi : \ C : a \to -a. \ \text{Note} : \theta \sim \theta + 2\pi \right)$

$\left(\text{By } S^1 \text{ compactifiation } \quad e^{i\phi} \sim e^{i\oint_{S^1} a} \right.$

$$S_E \sim \int \dot\phi^2 + i\frac{\theta}{2\pi}\int d\phi$$

We obtain the previous model w/ $\rightsquigarrow \theta = 2\pi S$)

The model has $U(1)^{[1]}$ symmetry

$$W(C) = e^{i\oint_C a} \mapsto e^{i\alpha}\, W(C).$$

To gauge this 1-form symmetry (i.e. center symmetry), introduce the $U(1)$ 2-form gauge field B.

Gauge trans.

$$B \longrightarrow B + d\lambda \quad \leftarrow U(1) \text{ gauge field}$$

We require that a transforms as

$$a \mapsto a + \lambda .$$

As a consequence, $W(C)$ is no longer gauge inv:

$$W(C) \mapsto e^{i\oint_C \lambda}\, W(C).$$

(instead, we find the gauge inv. surface operator $W(C)\, e^{-i\int_D B}$ w/ $\partial D = C$).

The minimal coupling gives

$$S_E = -\frac{1}{2g^2}\int (da - B)^2 + i\frac{\theta}{2\pi}\int (da - B) .$$

Assume $\theta = \pi n \quad (n = 0, \pm 1, \cdots)$.

System has C-symmetry.

$$Z[B] \xrightarrow{C:\ \theta \to -\theta} \int \mathcal{D}a \, e^{-\frac{1}{2g^2}\int (da-B)^2 + \frac{i\theta}{2\pi}\int (da-B)}$$

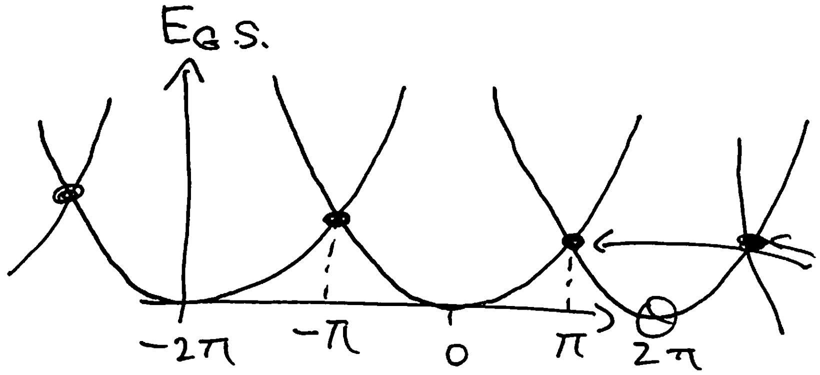
$$\times \, e^{-i \frac{\overbrace{(2\theta)}^{2\pi n}}{2\pi} \int (da - B)}$$

$$= Z[B] \, e^{in\int B}$$

Possible local counter term $e^{ik\int B}$ with $k \in \mathbb{Z}$.

$$Z[B]\, e^{ik\int B} \overset{C}{\longmapsto} \left(Z[B]\, e^{ik\int B}\right) \cdot e^{i(n-2k)\int B}$$

- $n = 0, 2, \cdots$ (i.e. $\theta = 0, 2\pi, \cdots$) $\Rightarrow$ $k = 2n$ eliminates the phase. No anomaly.
- $n = 1, 3, \cdots$ (i.e. $\theta = \pi, 3\pi, \cdots$) $\Rightarrow$ No k can eliminate the phase. $U(1)^{[1]}$ & $(\mathbb{Z}_2)_C$ anomaly.



doubly degenerate due to 't Hooft anomaly.

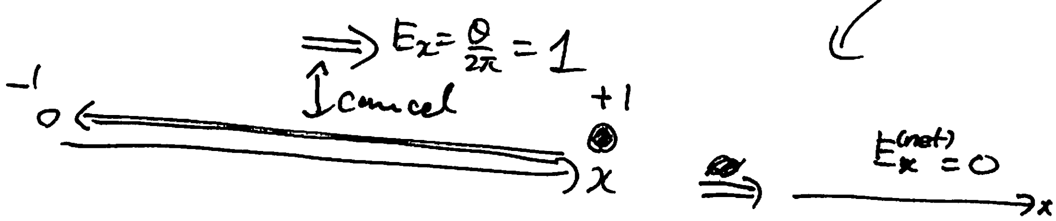
More on θ-terms in 2d Maxwell

S. Coleman : θ-angle in 2d = Background electric field
$$E_x = \frac{\theta}{2\pi}.$$

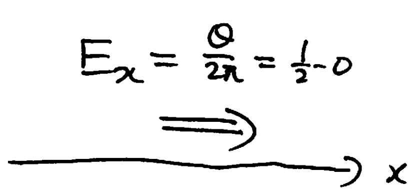
$$\therefore \quad \text{Energy} \sim E_x^2 = \left(\frac{\theta}{2\pi}\right)^2.$$

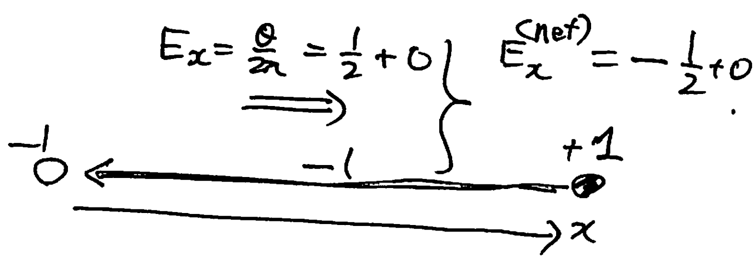
Q.) How can the energies @ $\theta = 0, 2\pi$ be the same?

A.) To cancel $E_x = 1$, put the charge ± 1 at infinities $x = \pm\infty$:

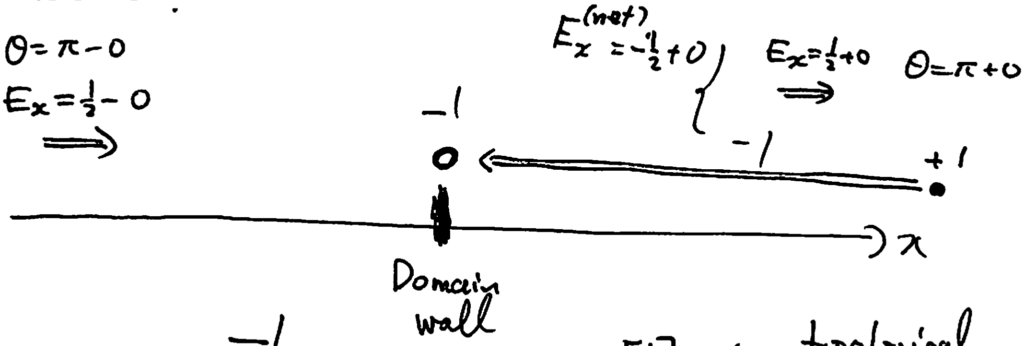


$\theta = \pi$ is doubly degenerate because





Domain wall.



DW is charged -1 under $U(1)^{[0]}$: $\leftarrow$ topological protection of domain wall excitations

Anomaly w/ 2-form gauge field wo/ 1-form symmetry

Consider 2d $\mathbb{CP}^1$ model: $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$. $z^\dagger z = 1$.

$$S_E = -\frac{1}{2g^2}\int |(d + i\underset{\substack{\uparrow \\ U(1)\text{ gauge field}}}{a})\vec{z}|^2 + \frac{i\theta}{2\pi}\int da .$$

Since z has charge 1 under $U(1)$, $U(1)^{[1]}$ is explicitly broken, i.e. No 1-form symmetry.

Global symmetry $\begin{cases} \theta \neq 0, \pi & G = \dfrac{\overset{\text{spin rotation}}{SU(2)}}{\mathbb{Z}_2} \\ \theta = 0, \pi & G = \dfrac{SU(2)}{\mathbb{Z}_2} \rtimes \underset{\substack{\text{charge conjugation} \\ z \to z^\dagger \\ a \to -a}}{(\mathbb{Z}_2)_C} \end{cases}$

Note: Although we have an $SU(2)$ invariance $z \mapsto U \cdot z$, the global symmetry is $SO(3) = \frac{SU(2)}{\mathbb{Z}_2}$, not $SU(2)$.
Any gauge inv. operators are $\mathcal{O}(x) \sim (z^\dagger(x))^n (z(x))^n$, and they are neutral under $\underset{z \mapsto -z}{\mathbb{Z}_2 \subset SU(2)}$.

↳ $SO(3)$ gauge field consists of
$\begin{cases} \cdot\ A: & SU(2) \text{ 1-form gauge field} \\ \cdot\ B: & \mathbb{Z}_2 \text{ 2-form gauge field.} \end{cases}$

At $\theta = 0$

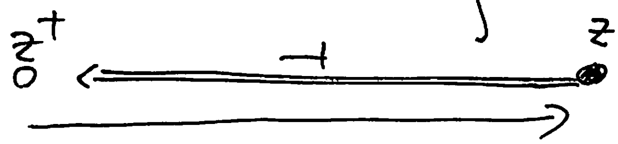
$$Z[A,B] \xmapsto{C} Z[A,B] \quad \text{No anomaly}$$

At $\theta = \pi$

$$Z[A,B] \xmapsto{C} Z[A,B]\, e^{i\int B} .$$

$\mathbb{Z}_2$ Anomaly of $SO(3) \rtimes \mathbb{Z}_2$.

$\theta = 2\pi$ $\qquad E_x = \frac{\theta}{2\pi} = 1 \Big\{ E_x^{(\text{net})} = 0.$



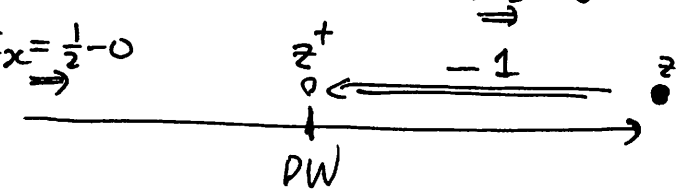
Boundary has projective rep. of $SU(2)/\mathbb{Z}_2$.

$\square, \square\square\square, \ldots$

Odd # of Young tableaux.

$\theta = \pi - 0$ $\qquad E_x = \frac{1}{2} + 0 \quad \theta = \pi + 0$

$E_x = \frac{1}{2} - 0$



$\mathbb{Z}_2$ topological projection of the DW excitation. //

To be more precise,

$\mathbb{C}P^1$ @ $\theta = 0$ $\Rightarrow$ Unique G.S. w/ mass gap
No anomaly.

$\mathbb{C}P^1$ @ $\theta = \pi$ $\Rightarrow$ $SU(2)_1$ WZW CFT.

$$\frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2} \supset \frac{SU(2)_V}{\mathbb{Z}_2} \times \underbrace{(\mathbb{Z}_2)_R}_{\substack{\parallel \\ (\mathbb{Z}_2)_C}}$$

has $\mathbb{Z}_2$ anomaly.

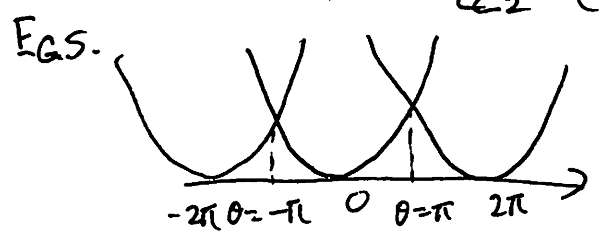
(Haldane conjecture)

This is generalized to $\mathbb{C}P^{N-1}$ with sym. $G = \frac{\overbrace{SU(N)}^{\text{spin}}}{\mathbb{Z}_N} \times (\mathbb{Z}_2)_C$.
(for even $N \geq 4$)

(cf. Komargodski, Sharon, Thorngren, Zhou
Komargodski, Sulejmanpasic, Ünsal)

$\mathbb{C}P^{N-1}$ @ $\theta = 0$ $\Rightarrow$ Unique G.S. w. mass gap
No anomaly

$\mathbb{C}P^{N-1}$ @ $\theta = \pi$ $\Rightarrow$ Double G.S. with mass gap
$\mathbb{Z}_2$ 't Hooft anomaly.



Other recent examples

3d $SU(N)_1$ Chern - Simons theory.

$$\frac{1}{4\pi}\int \mathrm{tr}\left(a\,da + \frac{2}{3}a^3\right).$$

Symmetry : $\mathbb{Z}_N^{[1]}$ symmetry.

$\longrightarrow$ B : $\mathbb{Z}_N$ 2-form gauge field.

Then, $Z_{cs}[B]$ is not gauge inv, but

$$Z_{cs}[B]\; e^{-i\underbrace{\frac{N}{4\pi}\int_{M_4} B\wedge B}_{\text{4d } \mathbb{Z}_N \text{ topological action.}}}$$

is gauge inv.

$\Rightarrow$ 3d $\mathbb{Z}_N$ topological order.

4d $SU(N)$ Yang - Mills at $\theta = \pi$.

$$-\frac{1}{2g^2}\int \mathrm{tr}\left[(da+a^2)^2\right] + i\,\frac{\overset{\pi}{\tilde{\theta}}}{8\pi^2}\int \mathrm{tr}\left[(da+a^2)^2\right].$$

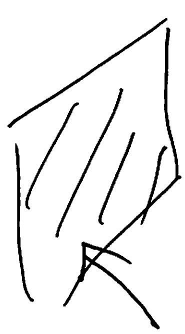
Symmetry : $\underbrace{\mathbb{Z}_N^{[1]}}_{} \rtimes (\mathbb{Z}_2)_{CP}$.

B : $\mathbb{Z}_N$ 2-form gauge field

$$Z[B] \xrightarrow{CP} Z[B]\; e^{i\underbrace{\frac{N}{4\pi}\int B\wedge B}_{\mathbb{Z}_2 \text{ anomaly.}}}$$

Same

$\langle G\tilde{G}\rangle < 0$



$\langle G\tilde{G}\rangle > 0$

$SU(N)_1$ Chern-Simons

Anomaly of S^1-compactified theory (Based on the work with T. Misumi, N. Sakai 1710.08923)

S^1-compactificati sometime provides a useful tool to study QFT

- It introduces an energy scale $E = \frac{1}{L}$.
- For asymptotically-free QFT, the weak-coupling analysis may become available if $\Lambda L \ll 1$.

Does this become a useful tool to study G.S. of QFT?

$\Rightarrow$ Vol. Indep. / Adiabatic continuity (Ünsal, ---)

Here, we have observed that

G.S. of QFT on $\mathbb{R}^d$ $\Leftarrow$ Constrained by 't Hooft anomaly.
(i.e. $(d+1)$-dim. top. action for anomaly inflow.)

Q.) Can this be continued to G.S. on $\mathbb{R}^{d-1} \times S^1$?

For this to be true, it's desirelle if

d-dim. Anomaly $\Rightarrow$ $(d-1)$-dim. Anomaly

to give the "same" constraint on the G.S.s.

This, however, is not as easy as it may sound.

Difficulty:

Quite often anomaly on $\mathbb{R}^d$ vanishes on $\mathbb{R}^{d-1}\times S^1$ with small S^1.

Counter example:

3d free Dirac : $\bar{\Psi}\gamma^i\partial_i\Psi$. $\underbrace{U(1)\times T}_{\exists\,\text{anomaly}}$.

$$Z[A] \xrightarrow{T} Z[A]\exp\left(\frac{i}{4\pi}\int A dA\right).$$

This ensures the masslessness.

$\Downarrow$ S^1-compactificati

$$\Psi(x^3+L) = -\Psi(x^3)$$

KK mass : $m_n = \frac{\pi}{L}(2n+1) \neq 0 \to$ Gapped unique vac.

i.e. No $U(1)\times T$ 't Hooft anomaly on $\mathbb{R}^2$.

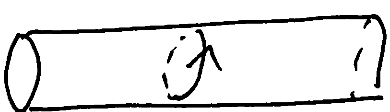
We can resolve this difficulty for

- Pure YM at $\theta=\pi$, $\underline{\text{2d } U(1) \text{ Maxwell @ } \theta=\pi}$, ...
  (i.e. Mixed anomaly w/ 1-form sym)
- $\underline{\mathbb{C}P^{N-1} \text{ at } \theta=\pi}$, 4d massless QCD, ...
  (i.e. Mixed anomaly w/ $PSU(N)=\frac{SU(N)}{\mathbb{Z}_N}$ flavor sym).

- 2d Maxwell @ $\theta=\pi$

$$Z[B] \xrightarrow{C} Z[B]\,e^{i\int B}$$

$\mathbb{R}^2 \to \mathbb{R}\times S^1$



Polyakov loop

$$e^{i\phi} = \mathrm{tr}\left(\mathcal{P}e^{i\oint_{S^1} a}\right)$$

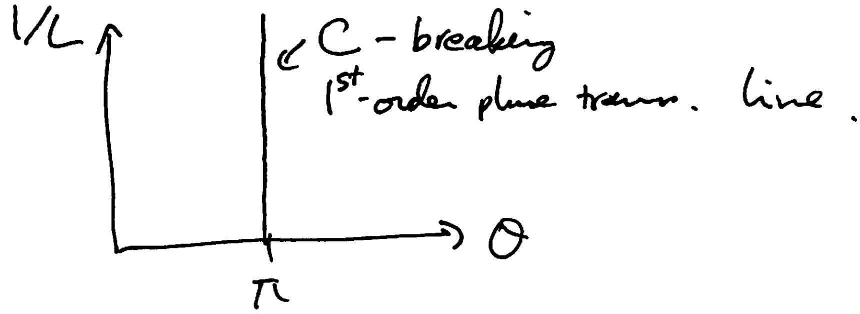
$U(1)^{[1]} \xrightarrow[S^1\text{-comp.}]{} U(1)^{[0]}$

$\phi \mapsto \phi+\alpha$

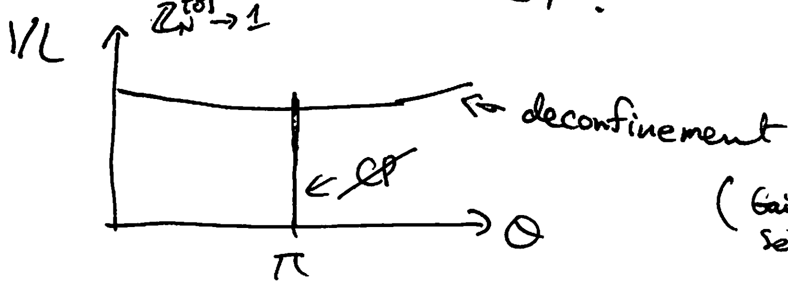
Introducing A : $U(1)^{[0]}$ gauge field.

In 2d language $B = A\wedge\frac{dx^2}{L}$. $\leadsto Z_{1d}[A] \xrightarrow{C} Z_{1d}[A]\,e^{i\int A\wedge\frac{dx^2}{L}} = Z_{1d}[A]\,e^{i\int A}$

$\Rightarrow$ At any size of S^1, the vac. @ $\theta = \pi$ are doubly degenerate.



(Similarly, in 4d YM, we can show the anomaly of $(\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[1]}) \rtimes (\mathbb{Z}_2)_{CP}$.



(Gaiotto, Kapustin, Komargodski, Seiberg))

• 2d $\mathbb{C}P^1$ @ $\theta = \pi$. $\quad S = \frac{1}{g^2}\int |(d + ia)\vec{z}|^2 + i\frac{\theta}{2\pi}\int da$.

Symmetry : $\underbrace{SO(3)}_{\frac{SU(2)}{\mathbb{Z}_2}} \rtimes (\mathbb{Z}_2)_C$

$SU(2)$ — A : $SU(2)$ gauge field

$\mathbb{Z}_2$ — B : $\mathbb{Z}_2$ 2-form gauge field.

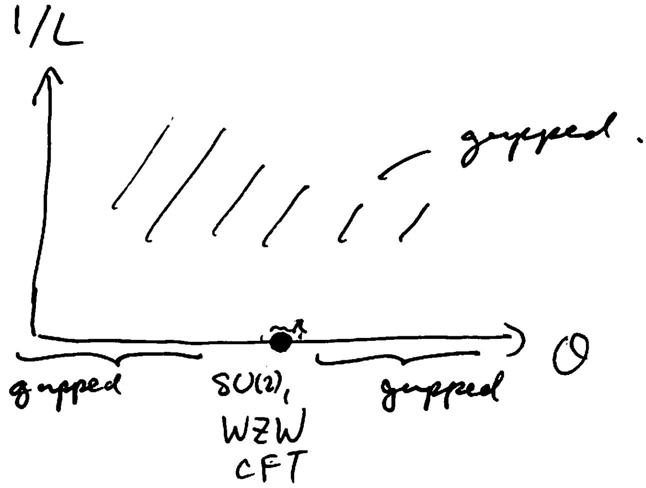
$$Z[A,B] \xrightarrow{C} Z[A,B]\, e^{i\int B}.$$

Compactification w/ P.B.C.

$\vec{z}(x^2 + L) = \vec{z}(x^2)$

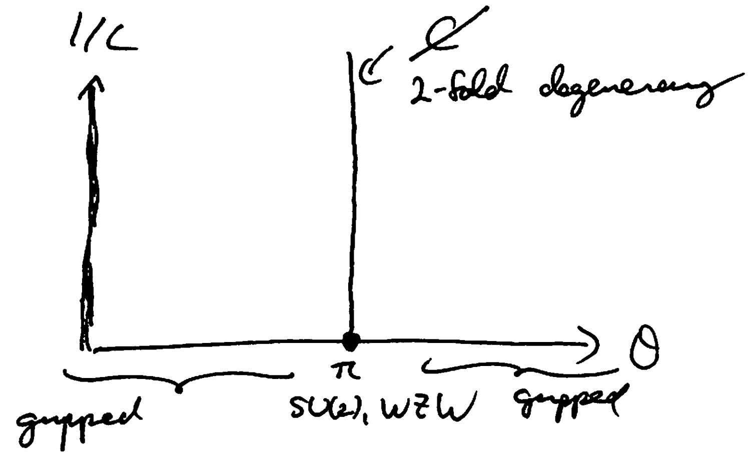
$\Rightarrow$ No anomaly.

Gapped unique vacuum.



Compactification with twisted B.C.

$$\vec{z}\,(x^2+L) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \vec{z}\,(x^2+L)$$



$\mathbb{Z}_2$ anomaly survives under twisted S^1-compactification

Polyakov loop phase

Reasoning: We have $\mathbb{Z}_2$ symmetry $\phi \mapsto \phi + \pi$

that has mixed anomaly with C.

(Kouno, et. al. '12~ --- in QCD-like; Cherman, et. al '17 in QCD)

Lag. : $\tilde{z}_1 = z_1$, $\tilde{z}_2 = e^{i\pi \frac{x^2}{L}} z_2$.

$$S = \int |(d+ia)\tilde{z}_1|^2 + \int |(d+ia+i\tfrac{\pi}{L}\delta_{\mu 2})z_2|^2 + i\frac{\theta}{2\pi}\int da$$

Therefore, $\mathbb{Z}_2$ trans.

$$\begin{cases} \phi \sim \dfrac{a_2}{L} \mapsto \phi + \pi \\ \tilde{z}_1 \leftrightarrow \tilde{z}_2 \end{cases}$$

is a symmetry.

$A := \mathbb{Z}_2$ gauge field. $\xrightarrow{\text{In 2d language}}$ $B = A \wedge \frac{dx^2}{L}$

Because both B, A should act on the Polyakov loop $e^{i\phi} = e^{i\oint a_2 dx^2}$.

$$Z_{t.b.c.}[A] \xrightarrow{C} Z_{t.b.c.}[A]\, e^{i\int_{\mathbb{R}\times S^1} B}$$

$$= Z_{t.b.c.}[A]\, \underbrace{e^{i\int_{\mathbb{R}} A}}_{\mathbb{Z}_2 \text{ anomaly}}$$